Appendix 7

The Prime Limit Theorem

Riemann's Explicit Formula* for the growth of $\pi(n)$, the number of primes $\leq n$, may be written

$$(0)\qquad \mathrm{R}(n) = \sum_{k=1}^{\infty} \frac{\mu(k)\mathrm{Li}(n^{\frac{1}{k}})}{k}$$

where μ is the Möbius function. It first struck me in January 1998 that Riemann had missed a much simpler way of expressing this result. Call $\mathrm{DR}(n) = \mathrm{R}((n+1)^2) - \mathrm{R}(n^2)$. Then $\mathrm{DR}(n)$ must be asymptotic to $n/\log n$.

I define weak convergence ~ to mean that the ratio of two functions approaches 1 as a common variable increases, but their difference does not regularly approach 0. If their difference regularly approaches 0, I define the convergence as strong or asymptotic $\approx$. Thus

$$(1)\qquad \mathrm{DR}(n) \approx n/\log n.$$

What makes (1) so useful is the closeness of the approach through the whole range of n. For example when $n = 10^4$, $n/\log n = 1085.7362...$ and the more-laborious computation of $\mathrm{DR}(n)$ using Riemann's formula shows no difference before the third decimal place.

It is now evident that we can adapt my formula (1) to achieve more simply what Riemann's does. Thus

$$(2)\qquad \mathrm{S}_t(n) = \frac{n}{(\sqrt{n}+t)^2}\left(\sum_{k=2}^{[\sqrt{n}+t]-1} \frac{k}{\log k} + \langle \sqrt{n}+t \rangle \frac{[\sqrt{n}+t]}{\log[\sqrt{n}+t]} \right)$$

where $<x>$ stands for 'fraction part of x' , i.e. $x - [\,x]$. I have used the fraction part of $\sqrt{n} + t$ to interpolate when n is not a natural square or t is not a whole number. This can introduce a small continuity error in a remote decimal place, of no consequence in a formula designed to predict cardinal numbers of primes. It is easy to prove that $\pi(n) \sim \mathrm{S}_t(n)$ for any finite value of t, but here we aim merely to match the performance of (0). Setting $t = 2$ will match evaluations of Riemann's formula to within ± 2 up to $n = 10^7$, after which t must be progressively reduced to allow for the diminution of the difference in (1).

Set $t = 1 - (\log n)/2$ and call $\mathrm{S}_t(n) = \mathrm{S'}(n)$ in this case. Then

$$(3)\qquad \mathrm{S'}(n) \bumpeq \mathrm{Li}(n).$$

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* Riemann, Bernhard, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.* Monatsberichte der Berliner Akademie, November 1859.

Thus although Riemann's formula cannot follow the curve of $\pi(n)$ when it crosses that of $\mathrm{Li}(n)$, mine can. A crossing is known to happen when n is around 6.2×10^{370}, so set $t = -426$ for this occasion. It is evident that the convergence of $\pi(n)$ to $S_t(n)$ must occur for any specific setting of t, positive or negative. Littlewood's theorem together with (3) assures us of covergence however large the negative value of t. For the positive side, choose t sufficiently large to make $S_t(n) = n/\log n$. Then $S_{t'}(n') = n'/\log n'$ when $t' \bumpeq t\sqrt{n'}/\sqrt{n}$, and this together with (4) assures us of convergence whatever the positive value of t. QED

The structure of Riemann's formula assures us that $\mathrm{DR}(n)$ must be almost identical with $\mathrm{Li}((n+1)^2) - \mathrm{Li}(n^2)$, so it remains only to show that this last expression is also strongly convergent to $n/\log n$, which is not difficult. Then the convergence to $\pi(n)$ of all the quantities listed here, including Riemann's, must follow.

My formula (1) implies that the function $n/\log n$ is an approximate indicator of the number of primes between n^2 and $(n+1)^2$ rather than between 0 and $n+1$ as was hitherto supposed. We can thus correct the Prime Number Theorem from

(4) $$\pi(n) \sim n/\log n$$

to the much more accurate

(5) $$\pi(n^2) \sim \sum_{k=2}^{n-1} \frac{k}{\log k} = S_0(n^2)$$

What is remarkable is that, in the last 200 years, Legendre, Gauss, Tchebychef, Riemann, Hadamard, de la Vallée Poussin, and the lesser mathematicians of the present century all misapplied $n/\log n$ to the wrong stretch of numbers.

This might have been because none of them could prove there were necessarily any primes between n^2 and $(n+1)^2$, so no one thought of asking how many there might be. Even I did not think of addressing this question until I had made an elementary proof, on 24th January 1998, of the existence of at least two primes between any two successive natural squares. After this the question of how many to expect became urgent, and I was able not only to prove that the expected number is near $n/\log n$, but also to ascertain sharp limits beyond which the actual number of primes between any n^2 and $(n+1)^2$ cannot deviate from $n/\log n$.

The total number $t(n)$ of primes between n^2 and $(n+1)^2$, where n is a natural number greater than 1, is within the limits

(6) $$A - (B-1) < t(n) < A + (B-1)$$

where A is $n/\log n$ and B is $A/\log A$.

Since my formula (6), which I call the Prime Limit Theorem, is not necessarily true unless n is one of the natural numbers 2,3,4,..., it further confirms the existence of a fundamental relationship between logarithms to the base e and the natural numbers, their squares, and the primes.

George Spencer-Brown, *1998 October 24*
Trinity College Cambridge

Correcting the Prime Number Theorem

After suggesting that the density of primes might be $1/\log x$, Gauss proceeded to make a conceptual blunder so enormous that it apparently stunned those who followed him into accepting it without question.

The chapter of horrors began when Gauss failed to say just *where* this hypothetical density was supposed to apply. The obvious place would have been at x, but instead Gauss proposed the theorem $\pi(x) \sim x/\log x$ (the present PNT), suggesting with uncharacteristic lack of precision that it must somehow apply indiscriminately to the whole stretch of positive numbers before x, resulting in the well-known answer that is, of course, absurdly too small.

The thought processes that led to Gauss's mistake are of some interest, since they were not confined to Gauss alone. But rather than consider why he made it, I propose, as simply as I can, to correct it.

The prime point density principle ppdp

Suppose the density of primes at the real number $x > 1$ *to be exactly* $\log^{-1}x$.*

I have defined density at a point, the real number x. To give it a verifiable meaning I shall have to expand the point into a restricted interval, say Δ, say of size $s \leq 2x$. We can now suppose prime density to be the ratio of primes to whole numbers in Δ.

Define $^\circ\pi(\Delta)$, say the prime degree or content of Δ, to be the number of primes in the interval Δ. We can now define Δ in terms of s and x so that $\Delta s(x)$ (say delta s of x) is the interval of size s whose midpoint is x. Interpret $s\log^{-1}x$ to be an *expectation* of the number of primes in $\Delta s(x)$. Assuming the truth of the ppdp, certain consequences must follow.

* Even this supposedly "well known" theorem is new, because no one had the discrimination to use the word 'exactly' in this context. To say 'approximately', as previous authors stated or implied, is simply wrong. If the theorem is not explicitly stated to be exact, the important consequences I draw from it will not follow.

Consequence 1

There exists an interval of size $s \leq 2x$ in which $°\pi(\Delta s(x))$ *and* $s\log^{-1}x$ *differ by less than* 1.

Consider $x = 500$, $s = 2x$. We establish an inequality, that for $x \geq 111^{1}/2$, $s = 2x$, $°\pi(\Delta s(x)) > s\log^{-1}x$, which manifests in this case as $°\pi(\Delta 1000(500)) = \pi(1000) = 168 > 1000\log^{-1}500 = 160.911...$. My Consequence 2 below assures us that there exists a critical value of $s < 2x$ where this inequality will be reversed. Let us try chopping 125 units off each end of the 1000, reducing s to 750. This happens to be about right (we do not need to look for anything more exact), $750\log^{-1}500 = 120.683...$ and the prime count in this smaller interval is $°\pi(\Delta 750(500)) = 120$. The expected reversal has just taken place, and we can call $s = 750$ a critical value for $x = 500$. Here we have what I call a perfect population of elements with a ratio of primes to integers that is within a fraction of the calculated density. Any further reductions of s will merely have the effect of taking samples from this perfect population. Since the procedure can be generalized for all x, I have, by this means, transformed what was, for 200 years, a notoriously inexact science into an unexpectedly exact one.*

Consequence 2

For every x, there exists a critical value of s where the curve of $°\pi(\Delta s(x))$ *either kisses or crosses that of* $s\log^{-1}x$.

We demonstrate this simply by remarking that if it were not so, the ppdp would be false.

What is now becoming clear is that I have constructed a set of five consequential theorems, the ppdp and its four consequences, which we may call the *elementals* of the system, any one of which may be taken as the initial basis of the system, and from the truth of which will follow the truth of all the other elementals in the system. Once we find such a system, all will fall into place and everything will confirm everything else.

The elementals of any system have no natural ordinal sequence. They have to be stated in some order, but whatever order we choose we find we have to refer to those we have not yet mentioned to explain those we have.

Consequence 3

The density of primes at x^k is exactly k^{-1} of their density at x. (*x and k real numbers,* $x > 1$, $k > 0$.)

*For even greater accuracy, we can average the prime counts above and below the point where the curve of $°\pi(\Delta s(x))$ first crosses that of $s\log^{-1}x$. We may even shift x to fractionally off centre so that prime counts do not jump by two at a time.

Corollary

All intervals $(\Delta ks(x^k))$ *have the same prime expectation.*

Call this the inverse power law. Having eliminated the natural logarithm, we can begin to see how the unifying principle of primality is of spectacular simplicity.

I decided to try out the ppdp in 1999 April 17, while rereading H M Edwards's book on Riemann's Zeta Function. The newness of the resulting mathematics, with the fact that I had to invent an unfamiliar technology to describe it, soon convinced me that nobody had seriously examined the idea before, even though it does seem such an obvious thing to do. The result has been that my C3 remained entirely unnoticed in all the previous literature. However obvious it may seem now that I have published it, it is certain no one was conscious of it before. I have known and worked with a dozen or more of the top number theorists for nearly half a century, and if any of us had thought of it he would have (a) mentioned it, (b) exploited it, and (c) published it. The law is far too *useful* to keep secret, once it has been seen.

Prime avenues

We can now apply the Corollary to my C3 to generate endless vistas of expanded intervals with the same prime expectation. Call them prime avenues.

x^k	Δ *of size ks*	$^\circ\pi(\Delta ks(x^k))$	x^k	Δ *of size ks*	$^\circ\pi(\Delta ks(x^k))$
350	$\pm\ 50$	16	$350^{2.5}$	± 125	14
$350^{1.5}$	$\pm\ 75$	16	350^{3}	± 150	18
350^{2}	± 100	18	$350^{3.5}$	± 175	17

Table 1. Prime avenue 350 : 100 (1 : 0.5 : 3.5). Expectation = $100 \log^{-1} 350$ = 17.0708673798
Average count per frame so far = 99/6 = 16.5

A colleague has just sent me an avenue 350 : 100 (1 : 1 : 50). The total count is 855 and the average is 17.1. The largest primes are 128-figure numbers and the nearest total to expectation would have been 854. It is not in fact necessary to calculate the expectation, because the prime counts will eventually supply it. Thus the natural logarithm can be eliminated not only practically, but also theoretically, from the theory of primes, since the primes themselves will always tell us what it is. In other words the primes, marshalled into avenues by my C3 Corollary, constitute yet another way of computing the number e.

Consequence 4

If m is the mean prime count in the avenue $x : s$, then $x^{m/s}$ is asymptotic to e.

This is the first time that any method has been devised for computing e from data supplied by the primes alone.* Of course, to get near the right answer, we would need to average hundreds more counts than my colleague did.

Supersaturated densities

We get prime densities > 1 when $x < e$, suggesting that in this region there are more primes than integers. When $x < 2$ there are apparently none of either, but the supersaturated densities my theorems demand can nevertheless be verified by raising x to a power k sufficient to make a number around which integers and primes can be counted. Do we regard the phenomenon as a computational convenience, or look for some other interpretation?

The accuracy of Riemann's formula

The known theorem that for $x \geq 11$, $\pi(2x) < 2\pi(x)$, implies that, for $x \geq 223/2$ and $s = 2x$, $s\log^{-1}x < {}^{\circ}\pi(\Delta s(x))$. If this inequality were not to reverse itself for some $s < 2x$, the ppdp would be false (compare C2), so in general we must expect $ks\log^{-1}x^k$, when ks is small in comparison with x^k, to overestimate very slightly the expectation of primes in $\Delta ks(x^k)$, but that this discrepancy will approach zero as k increases.

We can use this theorem as a highly-sensitive test, adjudging Riemann's formula to be accurate if it can be shown to incorporate this discrepancy. To check this we need to construct avenues of Riemann differences in place of prime counts. My uncorrected† expectation for an interval of size 7 whose midpoint is 10 is $7\log^{-1}10 = 3.040061373\ldots$, and $R(10 + 3\tfrac{1}{2}) - R(10 - 3\tfrac{1}{2}) = 2.7974\ldots$, a little below this expectation, as anticipated. Repeating the subtraction with $10^3 \pm 10\tfrac{1}{2}$, $10^6 \pm 21$, and $10^9 \pm 31\tfrac{1}{2}$ yields successive differences 2.9818, 3.0384, and 3.0401, which is extremely close to what my theorem predicts. So Riemann's formula is confirmed to be nearly perfect, subject only to small corrections as detailed on p 8.

*I have never understood the fuss about the so-called Prime Number Theorem, which has been to prove that a horribly inexact and basically wrong approximation to $\pi(x)$ is nevertheless convergent to it. I can invent a dozen or more such approximations in an afternoon, all better than Gauss's, but what is the point? We are aiming to discover what familiar object, if any, the sequence of primes resembles. I have shown beyond all question that this object is e, in other words, that the primes themselves are a *manifestation* of this number. A wrong formula will use e (they all did) to get a bad answer for the primes, but will not reconstitute e from the primes. A correct formula will work in both directions, as mine does. This is the only theoretical prerequisite. It is independent of empirical evidence, which *must* follow provided we guessed right about the object the primes resemble (= reassemble). Gauss and Legendre both guessed right, but were hopelessly inadequate to the task of constructing the requisite arithmetic.

† See p 8.

What Riemann could have asked himself, if he had approached the problem in a less roundabout way, is, can his formula be reversed to compute e from the primes? The answer is yes, but not as simply or as accurately as mine can.

It is not generally known that Ramanujan was also clever enough to rediscover Riemann's formula. In 1913 he sent it to Hardy, who replied: 'As regards your work about primes, the result is certainly wrong. Of this there is no doubt whatsoever.' No mathematician feels the need to be this emphatic*, unless he has taken up a false position.

Twenty three years later, Hardy discovered his mistake, having at last read the Riemann paper. Now realizing Ramanujan's formula was identical to Riemann's, but derived more simply, he called it, in lectures, the Riemann-Ramanujan formula, too late to benefit Ramanujan. But Hardy never admitted in print that he had changed his mind about it, so very few people are aware that Ramanujan had anything useful to say on this subject. – Author, reviewing this document in 1999 May 27.

Note (June 2005). My formula (2) may be replaced by its simpler equivalent

(2a) $$SB_t(n) = \frac{n}{(\sqrt{n}+t)^2}\left(\mathrm{Li}((\sqrt{n}+t-\tfrac{1}{2})^2)-1\tfrac{1}{2}\right)$$

This is much quicker for large n, where the summation in (2) can take many hours on a pocket calculator. Using (2a), $\mathrm{Li}(x)$ can be computed in a matter of seconds with Soldner's formula

where γ is Euler's constant and when necessary the absolute value of y is used to compute $\log y$.

* We can contrast the comment of my own tutor, J C P Miller, to whom I sent a theorem that was wrong to the point of idiocy: 'I don't think we can use this.'

† Note to page 6

How to correct the raw expectation $E(x,s) = s\log_e^{-1}x$, is of astonishing beauty, and caused me an 18-month search to find the answer. Contrary to (or in refinement of) what I have suggested so far, the primes relate not exactly to e but to a function $\sigma(x)$ that converges to e more slowly than the historical definition $(1+x^{-1})^x$.

Define $\sigma(x) = (1+x^{-1/2})^{x^{1/2}}$. Then the corrected expectation of primes in the interval of size s whose midpoint is x is

$$C(x,s) = s\log_{\sigma(x)}^{-1} x,$$

provided s is subcritical to x.

To make sense of this we must consider the fact that the Eulerian number e is the primary value of the exponential function "at infinity", and this is of no interest to the primes in the region of some number x less than infinity. At first I thought that what might be of interest to these primes would be the partial computation of e at x by the usual formula $(1+x^{-1})^x$, but this gave entirely wrong answers. The final piece of the puzzle remained hidden until I realized that only the primes not greater than $\sqrt{x}$ have any bearing on the names and places of the primes in the immediate neighbourhood of x. Thus the only function of any interest to these primes will be the partial computation of e at $\sqrt{x}$, and this is my function $\sigma(x)$.

We can now compare my corrected expectations, which are exact, with the Riemann differences in the avenue 10 : 7 instanced above. It will be seen that the Riemann differences have a tendency to wander from the line of exact expectations, especially in the early stages.

x	10	10^3	10^6	10^9
Spencer-Brown exact expectations	2.6415	2.9930	3.0385	3.0400
Riemann differences	2.7974	2.9818	3.0384	3.0401